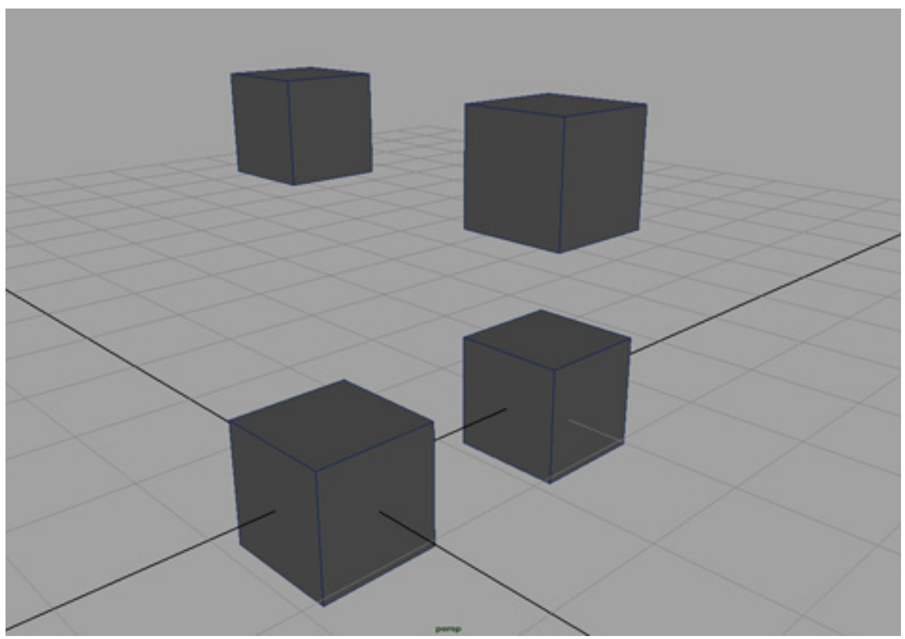
3D Transformations

COMP 770

Fall 2011

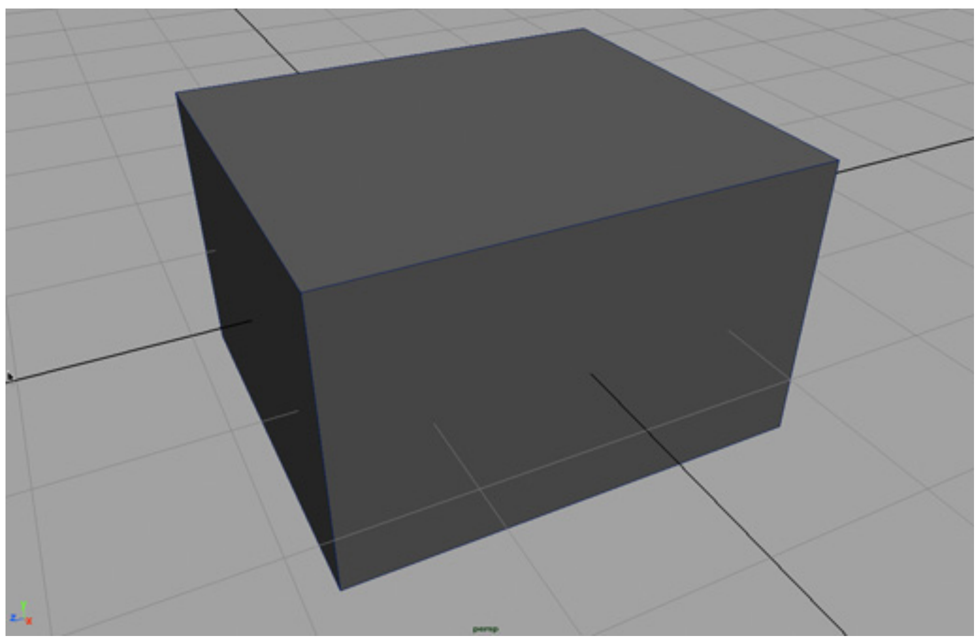
Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



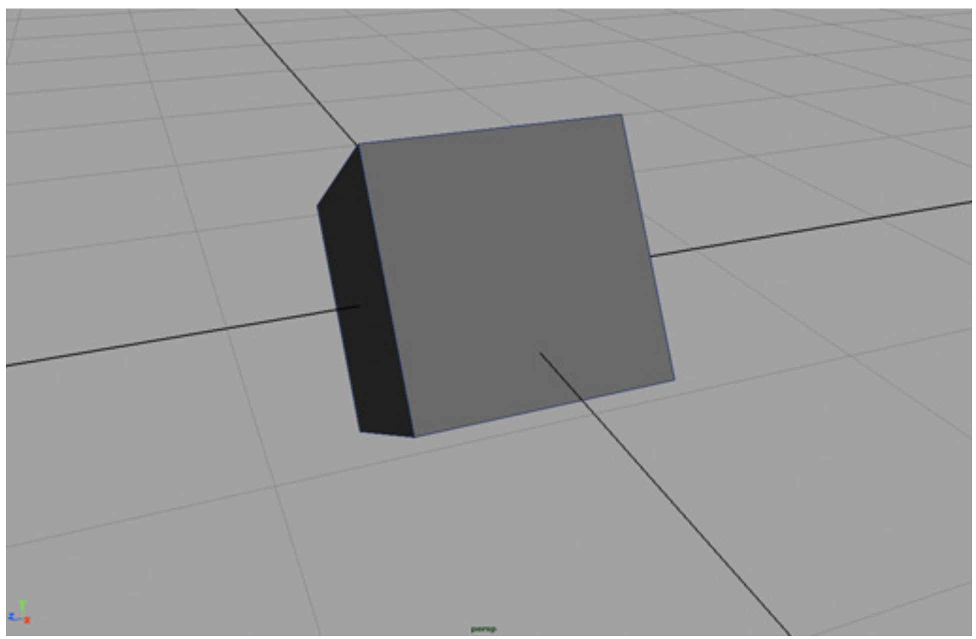
Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



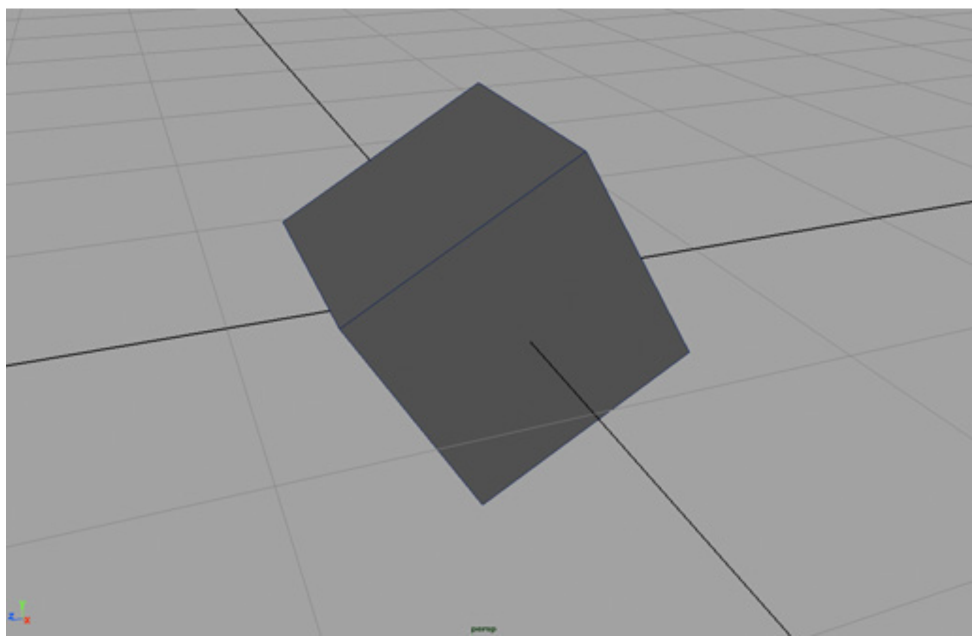
Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



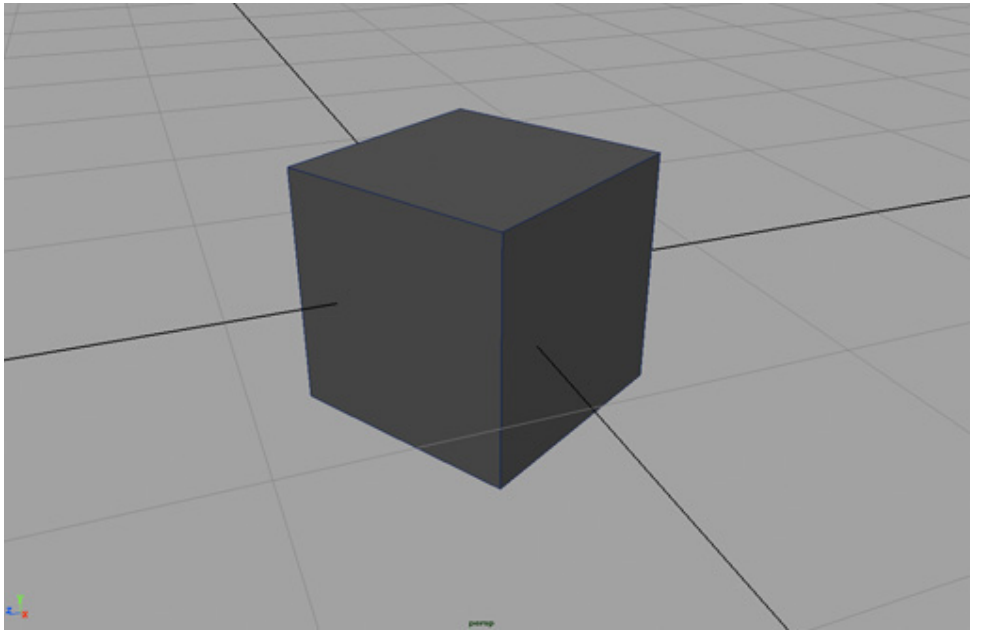
Rotation about *x* axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Rotation about y axis

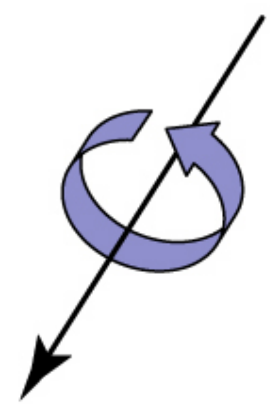
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



General rotations

- A rotation in 2D is around a point
- A rotation in 3D is around an axis
  - so 3D rotation is w.r.t a line, not just a point
  - there are many more 3D rotations than 2D
    - a 3D space around a given point, not just 1D

2D



3D

Specifying rotations

- In 2D, a rotation just has an angle
  - if it's about a particular center, it's a point and angle
- In 3D, specifying a rotation is more complex
  - basic rotation about origin: unit vector (axis) and angle
    - convention: positive rotation is CCW when vector is pointing at you
  - about different center: point (center), unit vector, and angle
    - this is redundant: think of a second point on the same axis...
- Alternative: Euler angles
  - stack up three coord axis rotations

Coming up with the matrix

- Showed matrices for coordinate axis rotations
  - but what if we want rotation about some random axis?
- Compute by composing elementary transforms
  - transform rotation axis to align with *x* axis
  - apply rotation
  - inverse transform back into position
- Just as in 2D this can be interpreted as a similarity transform

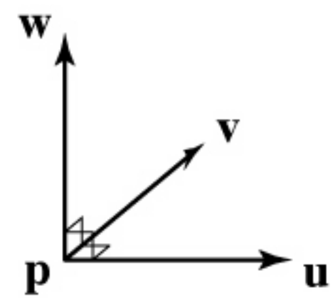
Building general rotations

- Using elementary transforms you need three
  - translate axis to pass through origin
  - rotate about *y* to get into *x*-*y* plane
  - rotate about *z* to align with *x* axis
- Alternative: construct frame and change coordinates
  - choose *p*, *u*, *v*, *w* to be orthonormal frame with *p* and *u* matching the rotation axis
  - apply similarity transform $T = F R_x(\theta) F^{-1}$

Orthonormal frames in 3D

- Useful tools for constructing transformations
- Recall rigid motions
  - affine transforms with pure rotation
  - columns (and rows) form right handed ONB
    - that is, an **o**rtho**n**ormal **b**asis

$$F = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Building 3D frames

- Given a vector **a** and a secondary vector **b**
  - The **u** axis should be parallel to **a**; the **u**–**v** plane should contain **b**
    - $\mathbf{u} = \mathbf{u} / ||\mathbf{u}||$
    - $\mathbf{w} = \mathbf{u} \times \mathbf{b}$; $\mathbf{w} = \mathbf{w} / ||w||$
    - $\mathbf{v} = \mathbf{w} \times \mathbf{u}$
- Given just a vector **a**
  - The **u** axis should be parallel to **a**; don't care about orientation about that axis
    - Same process but choose arbitrary **b** first
    - Good choice is not near **a**: e.g. set smallest entry to 1

Building general rotations

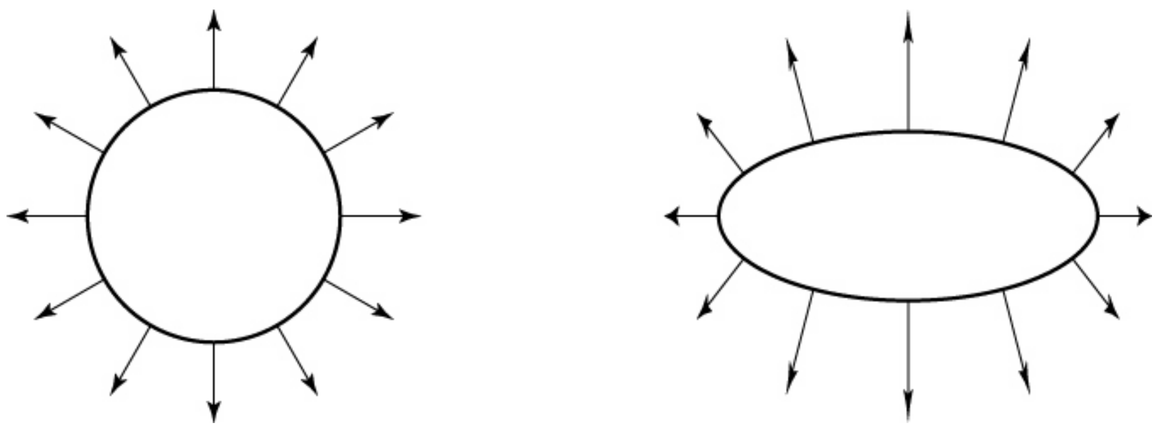
- Alternative: construct frame and change coordinates
  - choose p, u, v, w to be orthonormal frame with p and u matching the rotation axis
  - apply similarity transform $T = F\, R_x(\theta)\, F^{-1}$
  - interpretation: move to x axis, rotate, move back
  - interpretation: rewrite u-axis rotation in new coordinates
  - (each is equally valid)

Building transforms from points

- Recall2D affine transformation has 6 degrees of freedom (DOFs)
  - this is the number of “knobs” we have to set to define one
- Therefore 6 constraints suffice to define the transformation
  - handy kind of constraint: point **p** maps to point **q** (2 constraints at once)
  - three point constraints add up to constrain all 6 DOFs
    (i.e. can map any triangle to any other triangle)
- 3D affine transformation has 12 degrees of freedom
  - count them by looking at the matrix entries we’re allowed to change
- Therefore 12 constraints suffice to define the transformation
  - in 3D, this is 4 point constraints
    (i.e. can map any tetrahedron to any other tetrahedron)

Transforming normal vectors

- Transforming surface normals
  - differences of points (and therefore tangents) transform OK
  - normals do not



have: $\mathbf{t} \cdot \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$
want: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T X \mathbf{n} = 0$
so set $X = (M^T)^{-1}$
then: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T (M^T)^{-1} \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$